# Multivariate quantile impulse response functions

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- An important way to summarize the dynamics of macroeconomic data is to make use of a vector autoregressive (VAR) model. The VAR approach provides statistical tools for data description, forecasting, and structural inference to study rich dynamics in multivariate time-series models.
- Nevertheless, the use of a constant-coefficient model as representative of time-series models may not be adequate. These models cannot appropriately account for the presence of asymmetric and heterogeneous dynamic responses.
- Of particular interest is the asymmetric business cycle dynamics of economic variables, as the occurrence of asymmetries may call into question the usefulness of models with time invariant structures as means of modeling such series.
- Alternatives: nonlinear models, regimes, structural breaks, multivariate volatility models...
- ...or **quantile regression**.

- Quantile regression (QR) is a statistical method for estimating models of conditional quantile functions. This method offers a systematic strategy for examining how covariates influence the location, scale, and shape of the entire response distribution, thereby exposing a variety of heterogeneity in response dynamics.
- Koenker and Xiao (2006) QAR estimator applies QR models in time-series. Galvao, Montes-Rojas, and Park (2013) interpret the QR time-series framework as modeling the business cycle, where high (low) conditional realizations of a distributed lag model correspond to high (low) quantiles.
  Ex. AR(1) model:

$$E[Y_t|Y_{t-1}] = \alpha + \beta Y_{t-1}$$

*vs.*

$$Q_{Y_t}[\tau|Y_{t-1}] = \alpha(\tau) + \beta(\tau) Y_{t-1},\ \tau \in (0,1)$$

  where $Q_{Y_t}[\tau|Y_{t-1}]$ is the conditional quantile of $Y_t|Y_{t-1}$ (i.e., $F_{Y_t}^{-1}(\tau|Y_{t-1})$ for continuous cdf).
- Forecasting expected value vs. the full distribution (through quantiles).

- It is not possible to reproduce all "desirable properties" of scalar quantile regression in higher dimensions, so various proposals focus on achieving different sets of properties.
- Koenker (2005): "search for a satisfactory notion of multivariate quantiles has become something of a quest for the statistical holy grail in recent years."
- Consider a univariate random variable $Y$ with domain in $\mathcal{Y} \subseteq \mathbb{R}$ and distribution function $F_Y(y) := P(Y \leq y)$. Then the $\tau$th-quantile for $\tau \in (0,1)$ is defined as $Q_Y(\tau) := inf\{y \in \mathcal{Y} : \tau \leq F_Y(y)\}$. Note that if $F_Y(.)$ is continuous then $Q_Y(\tau) = F^{-1}(\tau)$.
- However, for $m$-variate random variable $Y$ with domain in $\mathcal{Y} \subseteq \mathbb{R}^m$, $inf\{y \in \mathcal{Y} : \tau \leq F_Y(y)\}$ is not unique.

# Directional quantiles

- Hallin, Paindaveine, and Šiman (2010) propose to analyze the distributional and quantile features of multivariate response variables using the directional quantiles notion of Chaudhuri (1996), Koltchinskii (1997), Wei (2008) and others. Further work by Paindaveine and Šiman (2011, 2012) and Fraiman and Pateiro-López (2012). Multivariate quantile analysis should be endowed with a magnitude and a direction.
- Carlier, Chernozhukov, and Galichon (2016) and Chernozhukov, Galichon, Hallin, and Henry (2015) propose a vector quantile regression (linear) model that produces a monotone map, in the sense of being a gradient of convex function.
- Montes-Rojas (2017) builds on directional quantiles and consider a model in which the orthonormal basis is fixed, i.e. a set of directions orthogonal to each other that span the domain of the dependent variable.
- The reduced form directional quantiles are defined as a fixed point of a system of directional quantiles.
- The solution maps $\mathcal{X} \times (0,1)^m \mapsto \mathcal{Y}$.

# My contribution

- Use directional QR to construct VARQ model for multivariate VAR.
- Construct and discuss forecasting procedures for the multivariate system.
- Introduce the idea of quantile paths, i.e., forecasting for different quantile configurations.
- Construct quantile impulse response functions (QIRFs).
- Empirical application: Evaluate the effect of monetary policy (shock in interest rate) on output and inflation (U.S.).

# The model

- Consider a $m-$dimensional process $\boldsymbol{Y}_t = (Y_{1t}, ..., Y_{mt})'$ with domain in $\mathcal{Y} \subseteq \mathbb{R}^m$. (endogeneous variables)
- Consider a $k-$dimensional process $\boldsymbol{X}_t$ with domain in $\mathcal{X} \subseteq \mathbb{R}^k$. (explanatory/control variables)
- Of particular interest is the case of the covariates generated by the $\sigma$-field generated by $\{\boldsymbol{Y}_s, s \leq t\}$ and all other information available at time $t$, denoted by $\mathcal{F}_t$. For that case the model becomes a vector autoregressive quantile (VARQ) model.
- For an autoregressive model of $p-$order then $\boldsymbol{X}_{t-1} = [\boldsymbol{Y}'_{t-1}, \boldsymbol{Y}'_{t-2}, ..., \boldsymbol{Y}'_{t-p}]'$ and $k = m \times p$.

- Quantiles are analyzed in terms of a quantile magnitude and a direction.
- Define $\boldsymbol{\tau} = (\tau_1, \tau_2, ..., \tau_m) \in (0,1)^m$ be a collection of quantile indexes.
- $\boldsymbol{\tau}$ factorizes into $\boldsymbol{\tau} \equiv \tau \boldsymbol{v}$ where $\tau = \|\boldsymbol{\tau}\| \in (0,1)$ (magnitude) and $\boldsymbol{v} \in \mathcal{V}^{m-1} \equiv \{\boldsymbol{v} \in \mathbb{R}^m : \|\boldsymbol{v}\| = 1\}$ (direction).
- $\tau$ represents a scalar quantile index;
- $\boldsymbol{v}$ is a $m - 1$-directional vector;

# Directional quantiles

- Let the vector $\boldsymbol{\tau}$ be an index on the open unit ball in $\mathbb{R}^m$ (deprived of the origin) $\mathcal{T}^m \equiv \{\boldsymbol{\tau} \in \mathbb{R}^m : 0 < \|\boldsymbol{\tau}\| < 1\}$. Our interest lies in defining and estimating

$$Q_{\boldsymbol{Y}_t|\boldsymbol{X}_t}(\boldsymbol{\tau}|\boldsymbol{X}_t) = \boldsymbol{B}(\boldsymbol{\tau})\boldsymbol{X}_t + \boldsymbol{A}(\boldsymbol{\tau}),$$

where $\boldsymbol{B}(\boldsymbol{\tau})$ is a $m \times k$ matrix of coefficients, $\boldsymbol{A}(\boldsymbol{\tau})$ is a $m \times 1$ vector of coefficients. Let $\boldsymbol{B}(\boldsymbol{\tau}) \equiv [\boldsymbol{B}_1(\boldsymbol{\tau})', \boldsymbol{B}_2(\boldsymbol{\tau})', ..., \boldsymbol{B}_m(\boldsymbol{\tau})']'$ where $\boldsymbol{B}_j(\boldsymbol{\tau})$, $j = 1, 2, ..., m$, are the corresponding $1 \times k$ vector of coefficients of the $j$th element in $\boldsymbol{Y}$.

- $Q$ is a map $\mathcal{X} \times \mathcal{T}^m \mapsto \mathcal{Y}$ and corresponds to our proposed definition of multivariate quantiles, which we will be defined as vector directional quantiles (VDQ).

# VARQ

- Define the **univariate** QR models for $j = 1, \ldots, m$

$$q_j(\tau_j|\boldsymbol{x}_{t-1}, \boldsymbol{y}_{-jt}) := Q_{Y_{jt}}(\tau_j|\boldsymbol{x}_{t-1}, \boldsymbol{y}_{-jt}) = \boldsymbol{c}_j(\tau_j)^\top \boldsymbol{y}_{-jt} + \boldsymbol{b}_j(\tau_j)^\top \boldsymbol{x}_{t-1} + a_j(\tau_j)$$

  [Note: This corresponds to a particular direction in the space $\mathcal{Y}$.]

- In order to construct the VARQ model define $Q_{\boldsymbol{Y}_t}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) := \{q_1(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}), \ldots, q_m(\boldsymbol{\tau}|\boldsymbol{x}_{t-1})\}^\top$ from the system of equations below:

$$\begin{cases} q_1(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) := \boldsymbol{c}_1(\tau_1)^\top q_{-1}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) + \boldsymbol{b}_1(\tau_1)^\top \boldsymbol{x}_{t-1} + a_1(\tau_1) \\ \vdots \qquad := \vdots \\ q_m(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) := \boldsymbol{c}_m(\tau_m)^\top q_{-m}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) + \boldsymbol{b}_m(\tau_m)^\top \boldsymbol{x}_{t-1} + a_m(\tau_m), \end{cases}$$

  where $\{\boldsymbol{c}_j(\tau_j)\}_{j=1}^m$ and $\{\boldsymbol{b}_j(\tau_j)\}_{j=1}^m$ are vectors of dimensions $(m-1) \times 1$ and $k \times 1$, respectively, and $\{a_j(\tau_j)\}_{j=1}^m$ are scalars.

- All the $m$-directions together correspond to an orthonormal basis. The solution is a fixed point or a simultaneous solution of all $m$ equations.

# VARQ

- Consider the following matrices based on the coefficients above: $\boldsymbol{C}(\boldsymbol{\tau}) := \{\boldsymbol{C}_1(\tau_1), \ldots, \boldsymbol{C}_m(\tau_m)\}^\top$ is an $m \times m$ matrix in which the $\{\boldsymbol{C}_j(\tau_j)\}_{j=1}^m$ $m \times 1$-dimensional vectors contain all the elements of the $m-1$ vector of coefficients $\{\boldsymbol{c}_j(\tau_j)\}_{j=1}^m$ augmented with a 0 in the corresponding $j$th component, $\boldsymbol{b}(\boldsymbol{\tau}) = \{\boldsymbol{b}_1(\tau_1), \ldots, \boldsymbol{b}_m(\tau_m)\}^\top$ is an $m \times k$ matrix, and $\boldsymbol{a}(\boldsymbol{\tau}) = \{a_1(\tau_1), \ldots, a_m(\tau_m)\}^\top$ is an $m \times 1$ vector.
- Then, the VARQ model is defined as

$$Q_{\boldsymbol{Y}_t}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1}) = \{\boldsymbol{I}_m - \boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\{\boldsymbol{b}(\boldsymbol{\tau})\boldsymbol{x}_{t-1} + \boldsymbol{a}(\boldsymbol{\tau})\} := \boldsymbol{B}(\boldsymbol{\tau})\boldsymbol{x}_{t-1} + \boldsymbol{A}(\boldsymbol{\tau}),$$

where $\boldsymbol{I}_m$ is the $m$-dimensional identity matrix, $\boldsymbol{B}(\boldsymbol{\tau}) := \{\boldsymbol{I}_m - \boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\boldsymbol{b}(\boldsymbol{\tau})$ and $\boldsymbol{A}(\boldsymbol{\tau}) := \{\boldsymbol{I}_m - \boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\boldsymbol{a}(\boldsymbol{\tau})$.

# VARQ

- Define the lag polynomials ($\boldsymbol{B}(\boldsymbol{\tau}, L)$) such that

$$\boldsymbol{B}(\boldsymbol{\tau})\boldsymbol{X}_t = \boldsymbol{B}(\boldsymbol{\tau}, L)\boldsymbol{Y}_t = \sum_{k=1}^{p} \boldsymbol{B}_{\cdot k}(\boldsymbol{\tau})L^k \boldsymbol{Y}_t$$

and

$$Q_{\boldsymbol{Y}_t}(\boldsymbol{\tau}|\boldsymbol{x}_{t-1} = \boldsymbol{X}_{t-1}) = \boldsymbol{B}(\boldsymbol{\tau}, L)\boldsymbol{y}_t + \boldsymbol{A}(\boldsymbol{\tau}),$$

where $\boldsymbol{y}_t$ denotes the values of $\boldsymbol{Y}_t$ to be used in the equation.

# Bivariate model with one exogenous covariate

- Consider the following motivating example of a bivariate model. Suppose a simple bivariate model with two response variables, $Y_1$ and $Y_2$, and an exogenous variable $X$.

$$Y_{1t} = \gamma_1 Y_{2t} + \beta_1 X_t + \alpha_1 + \epsilon_{1t},$$
$$Y_{2t} = \gamma_2 Y_{1t} + \beta_2 X_t + \alpha_2 + \epsilon_{2t},$$

  assuming that $X_t \perp (\epsilon_{1t}, \epsilon_{2t})$, $Y_{jt} \perp \epsilon_{(3-j)t}$, $j = 1, 2$, and $\epsilon_{1t} \perp \epsilon_{2t}$. Let $\boldsymbol{Y} = (Y_1, Y_2)^\top$ and $var(\epsilon_j) = \sigma_j^2$, $j = 1, 2$.
- We know that $(\gamma_1, \gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2)$ cannot be identified unless additional assumptions are made as in structural VAR models.

# Bivariate model with one exogenous covariate

- The conditional expectations $(E[Y_1|x], E[Y_2|x])$, i.e. the reduced form model, can be identified and consistently estimated by using the model

$$E[Y_{1t}|X_t = x] = \frac{\beta_1 + \gamma_1\beta_2}{1-\gamma_1\gamma_2}x + \frac{\alpha_1 + \gamma_1\alpha_2}{1-\gamma_1\gamma_2} = \tilde{\beta}_1 x + \tilde{\alpha}_1,$$
$$E[Y_{2t}|X_t = x] = \frac{\beta_2 + \gamma_2\beta_1}{1-\gamma_1\gamma_2}x + \frac{\alpha_2 + \gamma_2\alpha_1}{1-\gamma_1\gamma_2} = \tilde{\beta}_2 x + \tilde{\alpha}_2,$$

where $\tilde{\alpha}_j \equiv \frac{\alpha_j+\gamma_j\alpha_{3-j}}{1-\gamma_j\gamma_{3-j}}, \tilde{\beta}_j \equiv \frac{\beta_j+\gamma_j\beta_{3-j}}{1-\gamma_j\gamma_{3-j}}$, $j = 1, 2$ stands for the reduced form parameters.

# Bivariate model with one exogenous covariate

- Note that the reduced form can be found by a system of equations using

$$E\left[Y_{1t}|X_t = x\right] = \gamma_1 E[Y_{2t}|X_t = x] + \beta_1 x + \alpha_1 + E[\epsilon_{1t}|X_t = x],$$
$$E\left[Y_{2t}|X_t = x\right] = \gamma_2 E[Y_{1t}|X_t = x] + \beta_1 x + \alpha_2 + E[\epsilon_{2t}|X_t = x],$$

where $E[\epsilon_{3-j}|X_t = x] = 0, j = 1, 2$. In other words, the reduced form can be obtained by evaluating at the corresponding conditional expectations (where we are conditioning on $X$) only.

# Bivariate model with one exogenous covariate

- Consider now the conditional form model

$$E[Y_{1t}|Y_{2t},X_t] = c_1 Y_{2t} + b_1 X_t + a_1,$$

$$E[Y_{2t}|Y_{1t},X_t] = c_2 Y_{1t} + b_2 X_t + a_2,$$

and note that by adding an irrelevant endogenous variable to the model above produces another model for which in general $a_j \neq \alpha_j$, $b_j \neq \beta_j$, $c_j \neq \gamma_j$, $j = 1, 2$.

- The conditional model should be interpreted then as a biased structural system, provided that $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ may not be recovered.

# Bivariate model with one exogenous covariate

- Now consider the following system of equations:

$$y_1 \equiv E[Y_{1t}|Y_{2t} = y_2, X_t = x] = c_1 E[Y_{2t}|Y_{1t} = y_1, X_t = x] + b_1 x + a_1,$$

$$y_2 \equiv E[Y_{2t}|Y_{1t} = y_1, X_t = x] = c_2 E[Y_{1t}|Y_{2t} = y_2, X_t = x] + b_2 x + a_2,$$

and its solution

$$y_1 = \frac{b_1 + c_1 b_2}{1 - c_1 c_2} x + \frac{a_1 + c_1 a_2}{1 - c_1 c_2} = \tilde{b}_1 x + \tilde{a}_1,$$

$$y_2 = \frac{b_2 + c_2 b_1}{1 - c_1 c_2} x + \frac{a_2 + c_2 a_1}{1 - c_1 c_2} = \tilde{b}_2 x + \tilde{a}_2,$$

where $\tilde{a}_j = \frac{a_j + c_j a_{3-j}}{1 - c_j c_{3-j}}$ and $\tilde{b}_j = \frac{b_j + c_j b_{3-j}}{1 - c_j c_{3-j}}$, for $j = 1, 2$.

# Bivariate model with one exogenous covariate

**Proposition**

*Conditional and reduced form coincide. That is for* $j = 1, 2$,

$$\tilde{a}_j = \frac{a_j + c_j a_{3-j}}{1 - c_j c_{3-j}} = \frac{\alpha_j + \gamma_j \alpha_{3-j}}{1 - \gamma_j \gamma_{3-j}} = \tilde{\alpha}_j,$$

$$\tilde{b}_j = \frac{b_j + c_j b_{3-j}}{1 - c_j c_{3-j}} = \frac{\beta_j + \gamma_j \beta_{3-j}}{1 - \gamma_j \gamma_{3-j}} = \tilde{\beta}_j.$$

# Bivariate model with one exogenous covariate

- Can this be generalized to quantiles?

$$\begin{aligned} Q_1(\tau_1, y_2, x) &\equiv Q_{Y_1|Y_2,X}(\tau_1|y_2, x) = c_1(\tau_1)y_2 + b_1(\tau_1)x + a_1(\tau_1) \\ Q_2(\tau_2, y_1, x) &\equiv Q_{Y_2|Y_1,X}(\tau_2|y_1, x) = c_2(\tau_2)y_1 + b_2(\tau_2)x + a_2(\tau_2). \end{aligned}$$

- Each equation will be seen as a particular directional quantile, as in Hallin, Paindaveine, and Šiman (2010). As such, they provide useful information about the joint distribution of $(Y_1, Y_2)$. However, the parameters $(c_j, b_j, a_j), j = 1, 2$ do not have a structural interpretation.

# Bivariate model with one exogenous covariate

- Set now the system of equations to solve for $Q_1(\tau_1,\tau_2,x)$, $Q_2(\tau_1,\tau_2,x)$, defined as

$$\begin{aligned} Q_1(\tau_1,\tau_2,x) &\equiv Q_1\left(\tau_1, Q_2(\tau_2, Q_1(\tau_1,\tau_2,x),x),x\right) \\ &= c_1(\tau_1)Q_2(\tau_1,\tau_2,x) + b_1(\tau_1)x + a_1(\tau_1) \\ Q_2(\tau_1,\tau_2,x) &\equiv Q_2\left(\tau_2, Q_1(\tau_1, Q_2(\tau_1,\tau_2,x),x),x\right) \\ &= c_2(\tau_2)Q_1(\tau_1,\tau_2,x) + b_2(\tau_2)x + a_2(\tau_2) \end{aligned}$$

- Then the definition of VARQ is thus given by $(Q_1(\tau_1,\tau_2,x), Q_2(\tau_1,\tau_2,x))$:

$$\begin{aligned} Q_1(\tau_1,\tau_2,x) &= \frac{b_1(\tau_1)+c_1(\tau_1)b_2(\tau_2)}{1-c_1(\tau_1)c_2(\tau_2)}x + \frac{a_1(\tau_1)+c_1(\tau_1)a_2(\tau_2)}{1-c_1(\tau_1)c_2(\tau_2)} \\ &\equiv b_1(\tau_1,\tau_2)x + a_1(\tau_1,\tau_2) \\ Q_2(\tau_1,\tau_2,x) &= \frac{b_2(\tau_2)+c_2(\tau_2)b_1(\tau_1)}{1-c_1(\tau_1)c_2(\tau_2)}x + \frac{a_2(\tau_2)+c_2(\tau_2)a_1(\tau_1)}{1-c_1(\tau_1)c_2(\tau_2)} \\ &\equiv b_2(\tau_1,\tau_2)x + a_2(\tau_1,\tau_2) \end{aligned}$$

# Forecasting

## One-period ahead forecasting

- The VARQ model implicitly defines a one-period ahead forecasting method for the entire distribution of $\boldsymbol{Y}_{t+1}$ given all the information available at $t$.

$$Q_{\boldsymbol{Y}_{t+1}}(\boldsymbol{\tau}|\boldsymbol{x}_t) = Q_{\boldsymbol{Y}_{t+1}}(\boldsymbol{\tau}|\{\boldsymbol{y}_t, \boldsymbol{y}_{t-1}, ..., \boldsymbol{y}_{t-p}\}) = \boldsymbol{B}(\boldsymbol{\tau}, L)\boldsymbol{y}_{t+1} + \boldsymbol{A}(\boldsymbol{\tau}).$$

- Define thus $Q_1(\boldsymbol{\tau}|\boldsymbol{x}_t) = Q_{\boldsymbol{Y}_{t+1}}(\boldsymbol{\tau}|\boldsymbol{x}_t)$ as the one-period ahead forecast given all the information available at time $t$.

# Forecasting

## Two-periods ahead forecasting - quantile paths

- Consider now the two-periods ahead forecast, i.e. $t+2$, at quantiles $\boldsymbol{\tau}_2$.
- Note that this would depend on the response at $t+1$ and the implicit quantile $\boldsymbol{\tau}_1$. In turn then this would depend on both quantiles, $(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1)$. This is defined as a two-periods **quantile path**, where the collection of indexes correspond to a potential path of the system of endogenous variables. Then

$$Q_2\{(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1)|\boldsymbol{x}_t\} := Q[\boldsymbol{\tau}_2|\{Q_1(\boldsymbol{\tau}_1|\boldsymbol{x}_t), \boldsymbol{y}_t, \ldots, \boldsymbol{y}_{t-p+1}\}].$$

# Forecasting

## *h*-periods ahead forecasting - quantile paths

- In general the *h*-periods ahead forecast can be written as a function of the forecast of the previous quantiles

$$Q_h\{(\boldsymbol{\tau}_h,\dots,\boldsymbol{\tau}_1)|\boldsymbol{x}_t\} = \boldsymbol{B}(\boldsymbol{\tau}_h, L) Q_k\{(\boldsymbol{\tau}_k,\dots,\boldsymbol{\tau}_1)|\boldsymbol{x}_t\} + \boldsymbol{A}(\boldsymbol{\tau}_h),$$

  where $Q_k(.|.) = y_{t-k}$ if $L^k(t+h) \leq t$ and $(\boldsymbol{\tau}_k,\dots,\boldsymbol{\tau}_1)$, $k = 1,\dots,h-1$ is the $k$-periods **quantile path**.
- Then we can write

$$Q_h\{(\boldsymbol{\tau}_h,\dots,\boldsymbol{\tau}_1)|\boldsymbol{x}_t\} = \{\Pi_{k=1}^h \boldsymbol{B}(\boldsymbol{\tau}_k)\}\boldsymbol{x}_t + \sum_{k=1}^{h-1}\{\Pi_{j=k+1}^h \boldsymbol{B}(\boldsymbol{\tau}_j)\}\boldsymbol{A}(\boldsymbol{\tau}_k) + \boldsymbol{A}(\boldsymbol{\tau}_h),$$

# Forecasting

## Quantile paths

- This framework allows for forecasting different **quantile paths**.
- A canonical case is fixing $\boldsymbol{\tau}_i = (0.5, \ldots, 0.5)$ for all $i = 1, \ldots, h$, which corresponds to evaluating future values on the conditional **median** values of the endogenous variables. In general this procedure delivers similar estimates as the mean-based VAR forecasts.
- This procedure can be generalized for any $\boldsymbol{\tau}_i = (\tau, \ldots, \tau)$ for all $i = 1, \ldots, h$. In this case high values of $\tau$ correspond to the persistent occurrence of the $\tau$ conditional quantile in all endogenous variables.
- Moreover, we do not necessarily need the same $\tau$ for all endogenous variables equations. As an example in the empirical application we consider the 0.1 and 0.9 quantiles of output, while we keep the median for inflation and interest rate. As such, we are constructing a potential quantile path where output is either at the low or high end of the business cycle. See Galvao, Montes-Rojas, and Park (2013) for an interpretation of QR time-series models in terms of the business cycle.

# Forecasting

## Averaging intermediate steps

- Note however that if we are interested in the $h$-periods ahead forecast, this may not depend on the implicit quantile used for the $k$-step forecast, $k < h$. As such we could integrate out $\boldsymbol{\tau}_k$ by using $\boldsymbol{\tau}_k \sim IID\ U(0,1)^m$ for $k = 1, 2, ..., h-1$. Define $\bar{\boldsymbol{B}} := \mathrm{E}_{\boldsymbol{\tau}} \boldsymbol{B}(\boldsymbol{\tau})$ and $\bar{\boldsymbol{A}} := \mathrm{E}_{\boldsymbol{\tau}} \boldsymbol{A}(\boldsymbol{\tau})$. (Note that $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{A}}$ are not necessarily equal to the mean-based reduced form VAR coefficients.)
- Then,

$$Q_h(\boldsymbol{\tau}|\boldsymbol{x}_t) = \boldsymbol{B}(\boldsymbol{\tau})\bar{\boldsymbol{B}}^{h-1}\boldsymbol{x}_t + \boldsymbol{B}(\boldsymbol{\tau})\left\{\sum_{k=1}^{h-1} \bar{\boldsymbol{B}}^k \bar{\boldsymbol{A}}\right\} + \boldsymbol{A}(\boldsymbol{\tau}).$$

- As $h \to \infty$, the long run prediction converges to

$$\lim_{h\to\infty} Q_h(\boldsymbol{\tau}|\boldsymbol{x}_t) = \boldsymbol{B}(\boldsymbol{\tau})(\boldsymbol{I} - \bar{\boldsymbol{B}})^{-1}\bar{\boldsymbol{A}} + \boldsymbol{A}(\boldsymbol{\tau}).$$

# Impulse response functions

- Our interest lies in evaluating the propagation of shocks of the m-variate process. (Identification of shocks comes from elsewhere.)
- We then compute the impulse response function by comparing the multivariate quantiles at $\boldsymbol{x}_t^{\delta} := (\boldsymbol{y}_t + \boldsymbol{\delta}, \boldsymbol{y}_{t-1}, ..., \boldsymbol{y}_{t-p})$ with those at $\boldsymbol{x}_t = (\boldsymbol{y}_t, \boldsymbol{y}_{t-1}, ..., \boldsymbol{y}_{t-p})$.
- Define the $\boldsymbol{\tau}$-quantile impulse response function (QIRF) at $t+1$ for a shock at time $t$, $\boldsymbol{\delta} \in \mathcal{Y} \subseteq \mathbb{R}^m$, as

$$\mathrm{Qirf}_1(\boldsymbol{\tau}, \boldsymbol{\delta}|\boldsymbol{x}_t) = Q_1(\boldsymbol{\tau}|\boldsymbol{x}_t^{\delta}) - Q_1(\boldsymbol{\tau}|\boldsymbol{x}_t) = \boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau})\boldsymbol{\delta},$$

where $Q_1$ is the one-period ahead forecast.

# Impulse response functions

- Consider now the IRF two-periods ahead, i.e. $t+2$, at quantiles $\boldsymbol{\tau}_2$. Note that this would depend on the response at $t+1$ and the implicit quantile $\boldsymbol{\tau}_1$. In turn then this would depend on both quantiles, $(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1)$, defined as a quantile path.

$$\begin{aligned}\mathrm{Qirf}_{2(1)}\left\{(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1), \boldsymbol{\delta} | \boldsymbol{x}_t\right\} &= Q_2\left\{(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1) | \boldsymbol{x}_t^{\boldsymbol{\delta}}\right\} - Q_2\left\{(\boldsymbol{\tau}_2, \boldsymbol{\tau}_1) | \boldsymbol{x}_t\right\} \\ &= \begin{cases} (\boldsymbol{B}_{\cdot 2}(\boldsymbol{\tau}_2) + \boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau}_2)\boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau}_1))\boldsymbol{\delta} & p > 1 \\ \boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau}_2)\boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau}_1))\boldsymbol{\delta} & p = 1 \end{cases}.\end{aligned}$$

- Note however that if we are interested in the two-periods ahead forecast, this may not depend on the implicit quantile used for the one-step forecast. As such we could integrate out $\boldsymbol{\tau}_1$ by using $\boldsymbol{\tau}_1 \sim U(0,1)^m$. Then define

$$\begin{aligned}\mathrm{Qirf}_2(\boldsymbol{\tau}, \boldsymbol{\delta} | \boldsymbol{x}_t) &= Q_2(\boldsymbol{\tau} | \boldsymbol{x}_t^{\boldsymbol{\delta}}) - Q_2(\boldsymbol{\tau} | \boldsymbol{x}_t) \\ &= \begin{cases} (\boldsymbol{B}_{\cdot 2}(\boldsymbol{\tau}) + \boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau})\bar{\boldsymbol{B}}_{\cdot 1})\boldsymbol{\delta} & p > 1 \\ \boldsymbol{B}_{\cdot 1}(\boldsymbol{\tau})\bar{\boldsymbol{B}}_{\cdot 1}\boldsymbol{\delta} & p = 1 \end{cases}.\end{aligned}$$

# Impulse response functions

- This procedure above can be generalized for $h$-periods ahead IRFs, by defining

$$\mathrm{Qirf}_{h(h-1,\dots,1)}\left\{(\boldsymbol{\tau}_h, \boldsymbol{\tau}_{h-1}, \dots, \boldsymbol{\tau}_1), \boldsymbol{\delta}|\boldsymbol{x}_t\right\}$$

$$= Q_h\left\{(\boldsymbol{\tau}_h, \boldsymbol{\tau}_{h-1}, \dots, \boldsymbol{\tau}_1)|\boldsymbol{x}_t^{\delta}\right\} - Q_h\left\{(\boldsymbol{\tau}_h, \boldsymbol{\tau}_{h-1}, \dots, \boldsymbol{\tau}_1)|\boldsymbol{x}_t\right\},$$

  for a given *path* of multivariate quantiles $(\boldsymbol{\tau}_h, \boldsymbol{\tau}_{h-1}, \dots, \boldsymbol{\tau}_1)$ and shock $\boldsymbol{\delta}$ at time $t$.

- $$\mathrm{Qirf}_h(\boldsymbol{\tau}, \boldsymbol{\delta}|\boldsymbol{x}_t) = Q_h(\boldsymbol{\tau}|\boldsymbol{x}_t^{\delta}) - Q_h(\boldsymbol{\tau}|\boldsymbol{x}_t),$$

  when we integrate out the previous periods that were constructed by iterations. (This is different from the mean-based VAR analysis. In this case, by using the iterated expectations property, the effect on $h$ periods ahead is the result of the conditional expectations in the previous periods.)
- In the long run the QIRF for $h \to \infty$ becomes 0 for stationary models.

# Impulse response functions - local projections

- A robust model for constructing IRFs is based on Jordà (2005) local projections method. The central idea consists in estimating local projections at each period of interest (i.e., $t + h$) rather than extrapolating into increasingly distant horizons from a given model, as it is done with VAR.
- The advantages of local projections are numerous:
  1. they can be estimated by simple regression techniques;
  2. they are more robust to misspecification;
  3. joint or point-wise analytic inference is simple;
  4. they easily accommodate experimentation with highly nonlinear and exible specications that may be impractical in a multivariate context.

# Impulse response functions - local projections

- This framework can be easily implemented in a VARQ context by a modeling the VDQ model of $\boldsymbol{Y}_{t+h}$ at each horizon $h = 1, 2, \ldots$ given all the information available at $t$, that is, all the lags of the endogenous variables up to $t$ (plus exogenous variables if any)

$$Q_h^{lp}(\boldsymbol{\tau}|\boldsymbol{x}_t) := Q_{\boldsymbol{Y}_{t+h}}(\boldsymbol{\tau}|\boldsymbol{x}_t) = \boldsymbol{B}_h(\boldsymbol{\tau})\boldsymbol{x}_t + \boldsymbol{A}_h(\boldsymbol{\tau}).$$

  Note that in this case we require to solve a different set of coefficients for each horizon $h$, which in fact involves directional QR models involving regressing $\boldsymbol{Y}_{j,t+h}$ on $\boldsymbol{Y}_{-j,t+h}$ and $\boldsymbol{x}_t$, for $j = 1, \ldots, m$.
- Then we could construct the QIRFs as

$$\mathrm{Qirf}_h^{lp}(\boldsymbol{\tau}, \boldsymbol{\delta}|\boldsymbol{x}_t) = Q_h^{lp}(\boldsymbol{\tau}|\boldsymbol{x}_t^{\delta}) - Q_h^{lp}(\boldsymbol{\tau}|\boldsymbol{x}_t).$$

- While this is an important alternative for prediction, it does not allow us to study quantile paths. That is, intermediate realizations of the random variables, i.e., for $h-1, h-2, \ldots, 1$, are implicitly evaluated at the mean-based values.

# Effect of monetary policy

- We estimate a three-variable (output gap, inflation, Fed Funds rate) VAR(1) model using U.S. quarterly data from 1980q1 to 2010q1 (121 quarters). This simple framework corresponds to the three-variable framework of New Keynesian model rational expectations model of Cho and Moreno (2004, 2006) and Jordà (2005), among others.
- The output gap is generated by the first-difference of the Hodrick-Prescott linear filter with linear trend, using the logarithm of the Gross National Product, 1996 constant prices (source: Federal Reserve Bank of St. Louis), denoted $y_t$.
- The inflation rate is the log first-difference of the GDP deflator, seasonally adjusted (source: Federal Reserve Bank of St. Louis), denoted $\pi_t$.
- The Fed Funds rate is the monetary policy instrument (source: Board of Governors of the Federal Reserve System), denoted $r_t$, and corresponds to the first-difference of the 3-months Treasury Bill rate (end of the quarter). The reason we use the first-difference of the interest rate is that over the period of analysis it shows a negative trend and we cannot reject it has a unit root.
- For this case then $\boldsymbol{Y}_t = (y_t, \pi_t, r_t)$.

# Effect of monetary policy

- We follow the Cholesky identification procedure in Christiano, Eichenbaum, and Evans (1996), using the residuals from a VAR model where we assume the standard ordering:
  - $r$ has no contemporaneous effect on $y$ and $\pi$;
  - $\pi$ has an effect on $r$ but not on $y$; and
  - $y$ affects both $\pi$ and $r$. This implies that shocks to the Fed Funds rate has no contemporaneous effect on the other economic variables.
- Then we evaluate the effect of a shock in $r$, calculated as the standard deviation of this structural shock, on output gap and inflation (also standardized by the standard deviation of their corresponding structural shocks).

Figure: Series 1980q1-2010q1

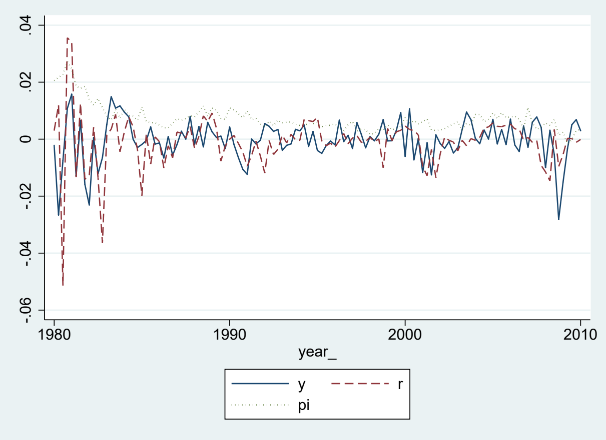
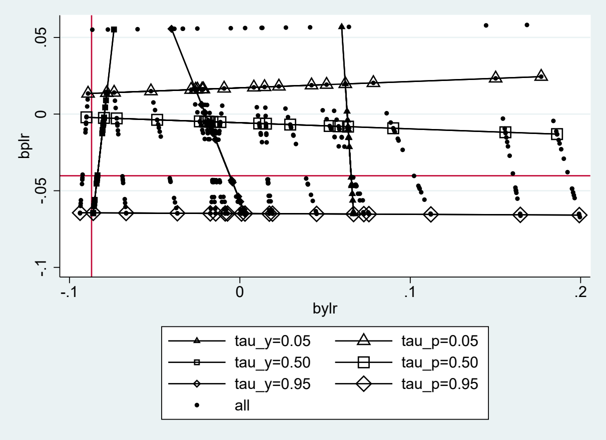


Table: Summary statistics for the series 1980q1-2010q1

| Variable | Obs | Mean | Std. Dev. | Min | Max |
|---|---|---|---|---|---|
| $y$ | 121 | -0.00034663 | 0.0072249 | -0.0281541 | 0.0158011 |
| $\pi$ | 121 | 0.0072152 | 0.0047343 | -.0016704 | 0.0272542 |
| $r$ | 121 | -0.00962 | 0.0093464 | -0.0514 | 0.0356 |
| Correlations $(y_t, \pi_t, r_t)$ | | | | | |
| Variable | $y_t$ | $\pi_t$ | $r_t$ | | |
| $y_t$ | 1.0000 | | | | |
| $\pi_t$ | -0.1068 | 1.0000 | | | |
| $r_t$ | 0.3831 | 0.1275 | 1.0000 | | |
| Correlations $(y_t, \pi_t, r_t)$ mean-based VAR residuals | | | | | |
| Variable | $y_t$ | $\pi_t$ | $r_t$ | | |
| $y_t$ | 1.0000 | | | | |
| $\pi_t$ | -0.0023 | 1.0000 | | | |
| $r_t$ | 0.3329 | 0.0593 | 1.0000 | | |

Figure: VARQ coefficients for $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$ and $\tau_r = 0.50$



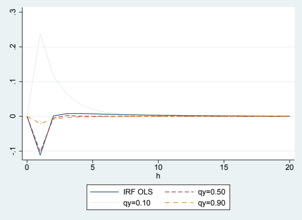

Notes: Vertical and horizontal lines correspond to the mean-based VAR effects.

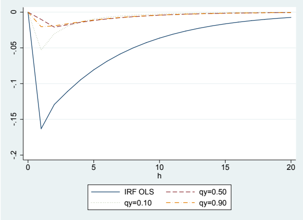
Table: VAR system stability - Modulus of eigenvalues of $\boldsymbol{B}(\boldsymbol{\tau})$

| Model | eigen 1 | eigen 2 | eigen 3 |
|---|---|---|---|
| $VAR-OLS$ | 0.853 | 0.152 | 0.067 |
| $VARQ(\tau_y = 0.5, \tau_\pi = 0.1, \tau_r = 0.5)$ | 0.669 | 0.131 | 0.131 |
| $VARQ(\tau_y = 0.1, \tau_\pi = 0.5, \tau_r = 0.5)$ | 0.813 | 0.535 | 0.054 |
| $VARQ(\tau_y = 0.5, \tau_\pi = 0.5, \tau_r = 0.5)$ | 0.818 | 0.145 | 0.145 |
| $VARQ(\tau_y = 0.5, \tau_\pi = 0.9, \tau_r = 0.5)$ | 0.984 | 0.153 | 0.153 |
| $VARQ(\tau_y = 0.9, \tau_\pi = 0.5, \tau_r = 0.5)$ | 0.820 | 0.285 | 0.058 |

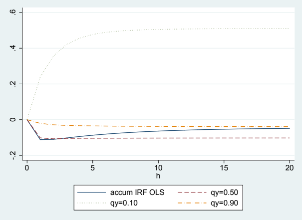
Output gap | Inflation

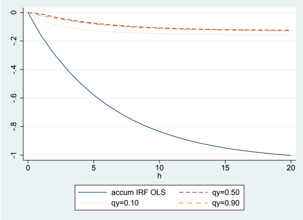
QIRF




Accumulated QIRF

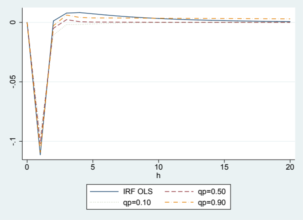


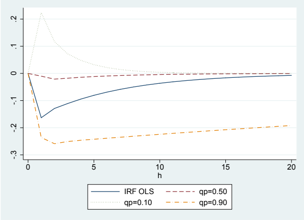




Notes: QIRF on output gap and inflation of a std.dev. shock in $r_t$ for $\tau_y \in \{0.10, 0.50, 0.90\}$, $\tau_\pi = 0.50$ and $\tau_r = 0.50$.

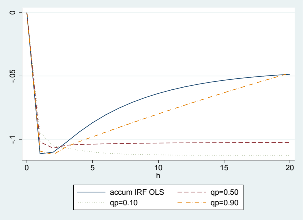
Output gap | Inflation

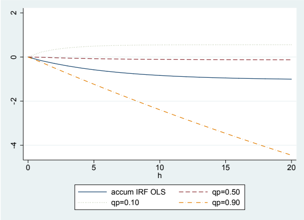
QIRF




Accumulated QIRF




Notes: QIRF on output gap and inflation of a std.dev. shock in $r_t$ for $\tau_p \in \{0.10, 0.50, 0.90\}$, $\tau_y = 0.50$ and $\tau_r = 0.50$.

# Further research ideas

- The model can be extended to nonlinear QR models. Fix point solution to a nonlinear system.
- Multivariate density forecasting. Consider a grid of G $m$-quantile indexes $\{\boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_G}\}$, then $\{Q_{\boldsymbol{Y}_t}(\boldsymbol{\tau_1}|\boldsymbol{X}_{t-1}), \ldots, Q_{\boldsymbol{Y}_t}(\boldsymbol{\tau_G}|\boldsymbol{X}_{t-1})\}$ can be used to construct $\{f_{\boldsymbol{Y}_t}(\boldsymbol{\tau_1}|\boldsymbol{X}_{t-1}), \ldots, f_{\boldsymbol{Y}_t}(\boldsymbol{\tau_G}|\boldsymbol{X}_{t-1})\}$ density points for $f : \mathcal{Y} \to \mathbb{R}$.
- Structural VARQ. Cholesky decomposition or other identification strategies for different quantile indexes.
- Quantile path analysis as an alternative to structural breaks.

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